AD699921

NORTH CAROLINA STATE UNIVERSITY

RALEIGH, NORTH CAROLINA

SOME ONE-DIMENSIONAL SOLUTIONS OF NONLINEAR WAVES OF A RATE-SENSITIVE, ELASTOPLASTIC MATERIAL

E. E. Burniston T. S. Chang

Technical Report 70-1, January 1970

Materials Response Phenomena At High Deformation Rates

PREPARED FOR
OFFICE OF NAVAL RESEARCH
DEPARTMENT OF THE NAVY
CONTRACT N00014-68-A-0187

SPONSORED IN PART BY
ADVANCED RESEARCH PROJECTS AGENCY
ARPA ORDER NO. 1090

40

NORTH CAROLINA STATE UNIVERSITY
Raleigh, North Carolina

SOME ONE-DIMENSIONAL SOLUTIONS OF NONLINEAR WAVES OF A RATE-SENSITIVE, ELASTOPLASTIC MATERIAL

E. E. Burniston T. S. Chang

Technical Report 70-1, January 1970

Prepared for

Office of Naval Research
Contract N00014-68-A-0187

1 September 1967 - 31 August 1972

under a project entitled

MATERIALS RESPONSE PHENOMENA AT HIGH DEFORMATION RATES

sponsored in part by

Advanced Research Projects Agency
ARPA Order No. 1090

ABSTRACT

Two classes of closed form solutions of one-dimensional, nonlinear waves of a rate-sensitive, elastoplastic material are reported. One class of these solutions is self-similar and the other class consists of constant speed propagations. Applications of these solutions to unsteady motions behind propagating discontinuities are also considered.

ACKNOWLEDGEMENT

This investigation was partially supported by the Advanced Research Projects Agency of the Department of Defense and monitored by the Office of Naval Research under Contract Number N00014-68-A-0187, and partially supported by the National Aeronautics and Space Administration under Grant Number NASA-NGL34-002-084. A portion of this work was completed when one of us (T. S. C.) was visiting the Center for the Application of Mathematics, Lehigh University. The authors are indebted to Professors R. S. Rivlin, E. Varley, and Y. Horie for their comments and suggestions.

TABLE OF CONTENTS

LIST OF FIGURES

I. INTRODUCTION

The purpose of this report is to discuss two interesting classes of closed form solutions of one-dimensional, unsteady motion of a rate-sensitive, elasto-plastic material. One class of these solutions is self-similar and is deduced from the invariant theorems of continuous groups of transformations. This class of unsteady motion is governed by a single, first-order, nonlinear, ordinary differential equation of the Riccati type and closed form solutions in terms of elementary functions are obtained under special circumstances. If the material in consideration possesses the additional property of instantaneous linear elasticity [1] under "high rate" of straining, it may be demonstrated that one of these self-similar solutions can be used to describe the dispersed nonlinear wave motion behind a propagating shockfront into an initially quiescent region.

The second class of solutions is obtained by searching for one-dimensional wave motions with constant speeds of propagation. These solutions are expressible as simple quadratures and closed form expressions can be obtained for specific constitutive relations. Such solutions represent non-characteristic propagations, i.e., they are not propagations of weak discontinuities or acceleration waves. It may be demonstrated, using the Poincaré-Bendixon theorem, that these solutions, in general, are not periodic. Assuming a sub-elastic, constant-speed, propagating discontinuity preceded by an elastic precurser with an unloading, relaxation zone, or a constant stress region, the nonlinear wave solution with a constant propagation speed equal to that of the discontinuity can be

used to describe the "unsteady"† motion behind the discontinuity.

One-dimensional rectilinear motion, in the strict sense, involves not just one spatial coordinate but also only one component of stress, strain, and particle velocity. Such a type of motion is typical of the propagation of longitudinal stress waves in thin straight rods when the lateral inertia effects of the rods can be neglected. For such a type of motion, only a one-dimensional stress-strain or constitutive relation is required. Various rate-sensitive, constitutive equations have been proposed in the literature and a comprehensive review of this subject can be found in Cristescu [2]. The solutions described in this report are obtained based on a model first proposed by Sokolovskii [3,4]. This model cannot, in general, be used to describe the structure or generation of shock waves [5,6]. In applying the solutions given in this report to the unsteady motions behind propagating shock layers or relaxation zones, additional material properties may have to be assumed within these regions.

II. MATHEMATICAL FORMULATION

One-dimensional motion may be described by a scalar deformation field,

$$\bar{x} = \bar{x}\,(\bar{X},\ \bar{t}) \quad , \tag{2.1}$$

where $\bar{x}$ is the instantaneous position coordinate at time $\bar{t}$ of a generic

---

†Such an "unsteady" motion, of course, becomes essentially steady for a moving observer following the propagating discontinuity.

particle whose position coordinate at $\bar{t} = 0$ was $\bar{X}$. The Lagrangian equation of motion and kinematic compatibility condition for rectilinear, one-dimensional motion are[†]

$$\rho \, \partial u/\partial \bar{t} = \partial \bar{\sigma}/\partial \bar{X} \quad , \tag{2.2}$$

$$\partial \varepsilon/\partial \bar{t} = \partial u/\partial \bar{X} \quad , \tag{2.3}$$

where $\bar{\sigma}$ is the longitudinal stress, and

$$u \equiv \partial \bar{x}/\partial \bar{t} \quad , \tag{2.4}$$

$$\varepsilon \equiv \partial \bar{x}/\partial \bar{X} - 1 \quad , \tag{2.5}$$

are the particle velocity and Lagrangian strain, respectively. The material is assumed to be initially unstressed and unstrained with a constant density ρ.

In this analysis, the material under consideration will be assumed to follow the special constitutive relation for a rate-sensitive, elastoplastic material generalized from a model suggested by Sokolovskii [3,4],

$$\partial \varepsilon/\partial \bar{t} = \partial u/\partial \bar{X} = E^{-1} \, \partial \bar{\sigma}/\partial \bar{t} + \gamma f \, (\bar{\sigma}/\sigma_o - 1) \, 1_+ \, (\bar{\sigma}/\sigma_o - 1) \quad , \tag{2.6}$$

where $f(\cdot)$ is a dimensionless C^1 function with $f(\eta) > 0$ for $\eta > 0$, $1_+(\cdot)$ is the Heaviside function, E is the modulus of elasticity which is assumed to be a

---

[†]See, e.g., Courant & Friedrichs, [7].

constant, σ_o is the static yield stress, and γ is a material constant. Thus, the material is assumed to have an elastic range with a constant modulus. In the plastic range, the dynamic yield stress is rate-sensitive. Strain-hardening effects are not included. Equation (2.6) includes the well-known models suggested by Cowper & Symonds [8], and Perzyna [9], as special cases. It is a special form of a more general constitutive equation suggested by Malvern [10].

Equations (2.2) and (2.6) are the basic equations describing the functions, $u(\bar{X}, \bar{t})$, $\bar{\sigma}(\bar{X}, \bar{t})$, [and $\epsilon(\bar{X}, \bar{t})$], characterizing the one-dimensional motions to be considered in this report. These equations may be combined into one single, second-order, nonlinear, hyperbolic, partial differential equation of the evolution type in dimensionless form as follows:

$$\beta(\partial^2\sigma/\partial x^2 - \partial^2\sigma/\partial t^2) = [1_+(\sigma)\, df(\sigma)/d\sigma + \delta(\sigma)\, f(0)]\, \partial\sigma/\partial t \quad , \qquad (2.7)$$

where

$$\sigma(x, t) \equiv \bar{\sigma}/\sigma_o - 1 \quad , \qquad (2.8)$$

is the dimensionless overstress,

$$x \equiv \alpha\bar{X} \quad , \qquad (2.9)$$

$$t \equiv \alpha c\bar{t} \quad , \qquad (2.10)$$

$$c \equiv \sqrt{E/\rho} \quad , \qquad (2.11)$$

$$\alpha \equiv \beta\gamma\sqrt{\rho E}/\sigma_o \quad , \qquad (2.12)$$

$\delta(\cdot)$ is the Dirac delta functional, and $\beta > 0$ is a dimensionless constant included here in the definition of α for convenience.

Materials described by the constitutive relation given in (2.6) probably cannot support shock layers or explain the generation of shockfronts. If a shock layer is dissipative, then generalized viscoelastic theories and constitutive relations such as those considered by Varley & Rogers [6], Coleman & Gurtin [11], Dunwoody & Dunwoody [12], and Pipkin [5], or further generalizations of these models, should be used to describe it. For a thin straight rod, the shock layer may be dispersive due to lateral deformation† instead of due to any dissipative mechanism. Such a shock transition may be described in terms of a low frequency, large rate of straining expansion of a three-dimensional deformation field similar to that considered by Parker & Varley [13]. In applying one of the self-similar motions described in this report to a nonlinear wave motion behind a propagating shockfront, it will be assumed that the rate of straining in the shock layer is high enough to allow the material to exhibit instantaneous linear elasticity††[1]. Thus, across such a shock layer, it will be assumed that

---

†The authors are indebted to Professor E. Varley for a discussion pertaining to this point.

††The range of rate of straining within which materials exhibit instantaneous elasticity varies from one material to another. There is usually an upper (and lower) cutoff point in rate of straining above (and below) which a material may have to be considered viscoelastic. The authors are indebted to Professor R. S. Rivlin for pointing this out to them.

$$[\bar{\sigma}] = E\,[\varepsilon] \quad , \tag{2.13}$$

where $[\chi]$ denotes the jump in value of χ across the shock layer, and the value of E will be assumed to be a constant and have the same value as the modulus of the elastic range of the constitutive relation given by (2.6).

From the Lagrangian equation of motion (2.2) and the kinematic compatibility condition (2.3), two additional jump conditions relating [u], and $[\varepsilon]$, can be deduced formally following a technique suggested by Courant & Friedrichs [7]. The results are:

$$\rho U\,[u] + [\bar{\sigma}] = 0 \quad , \tag{2.14}$$

$$U\,[\varepsilon] + [u] = 0 \quad , \tag{2.15}$$

where U is the propagation speed of the shockfront. Equations (2.14) and (2.15) can also be deduced from physical arguments directly. The jump conditions, (2.13)-(2.15), indicate that the speed of propagation of a shock layer of a material exhibiting instantaneous elasticity is

$$|U| = \sqrt{E/\rho} \quad , \tag{2.16}$$

which is, in fact, the same as the elastic speed of propagation of small disturbances.

In applying the constant speed solutions to the "unsteady" motion behind a propagating discontinuity which moves at a constant sub-elastic speed, it will be assumed that there is an elastic precurser and an unloading, relaxation

zone, or a constant stress region, ahead of the discontinuity. The details of the unsteady motion of a relaxation zone ahead of such a discontinuity may be very complicated and will not be considered in this report.

III. A CLASS OF SELF-SIMILAR SOLUTIONS

Cowper & Symonds [8] proposed, in 1957, a power law,

$$f(\sigma) = \sigma^{\delta} \quad , \tag{3.1}$$

where $\delta > 0$ is a dimensionless material constant, to describe the rate-sensitivity of perfectly plastic materials. This law seems to be quite adequate in approximating the dynamic responses of certain metallic alloys [14,15] under moderately high rates of straining. Recent investigators [16-20] have applied this model to impulsively loaded beams, rods, and plates. The class of self-similar solutions described in this report is based on the constitutive relation (2.6) and the special form of $f(\sigma)$ given by (3.1). Under these constitutive assumptions, Eq. (2.7) may be expressed as follows:

$$\phi\,(\sigma_{xx}, \sigma_{tt}, \sigma_t, \sigma;\, x,\, t) = 0 \quad , \tag{3.2}$$

where

$$\phi \equiv \sigma_{xx} - \sigma_{tt} - \sigma^{\delta-1}\,\sigma_t\, 1_+(\sigma) \quad , \tag{3.3}$$

and subscripts denote partial differentiation. The constant β which appeared in the definition of α in (2.12) has been replaced by the material constant δ.

Consider a one-parameter continuous group of transformations defined by

$$(X, T, \Sigma) = (bx, b^m t, b^n \sigma) \quad , \tag{3.4}$$

$$(\Sigma_{XX}, \Sigma_{TT}, \Sigma_T) = (b^{n-2}\sigma_{xx}, b^{n-2m}\sigma_{tt}, b^{n-m}\sigma_t) \quad , \tag{3.5}$$

where b is the parameter, and m,n are constants. It can be shown that for the special case of $m = 1$, $n = 1/(1 - \delta)$,

$$\phi\,(\sigma_{xx}, \sigma_{tt}, \sigma_t, \sigma; x, t) =$$

$$= b^{\frac{1-2\delta}{1-\delta}}\,\phi\,(\Sigma_{XX}, \Sigma_{TT}, \Sigma_T, \Sigma; X, T) \quad , \tag{3.6}$$

where it is assumed that $\delta \neq 1$. For $\delta = 1$, Eq. (3.2) is linear and the analytical solution has been discussed in detail by Malvern [10]. Thus, ϕ is a constant conformal invariant under the group defined by Eqs. (3.4) and (3.5) with $m = 1$, and $n = 1/(1 - \delta)$. According to a theorem proven by Morgan [21], the solution to Eq. (3.2) may be expressed in terms of a function $F(\xi)$ of an absolute invariant ξ of the transformation group defined by

$$(X, T) = (bx, bt) \quad . \tag{3.7}$$

The function $F(\xi)$ is an absolute invariant of the transformation group defined by

$$(X, T, \Sigma) = (bx, bt, b^{\frac{1}{1-\delta}}\sigma) \quad . \tag{3.8}$$

It will be straightforward to verify that

$$\xi = t/x \quad , \tag{3.9}$$

$$F(\xi) = x^{\frac{1}{\delta-1}} \sigma(x, t) \quad , \tag{3.10}$$

are absolute invariants of the groups defined by Eqs. (3.7) and (3.8), respectively. Thus, there exists a class of self-similar solutions to Eq. (3.2) of the form

$$\sigma = x^{\frac{1}{1-\delta}} F(\xi) \quad , \tag{3.11}$$

where ξ is given by (3.9).

Substituting Eq. (3.11) into Eq. (3.2) and using Eq. (3.3), a nonlinear, second-order ordinary differential equation results. For $\sigma > 0$, this equation is expressible as follows:

$$(\xi^2 - 1)F'' - \{[2\delta/(1-\delta)]\,\xi + F^{\delta-1}\}F' + [\delta/(1-\delta)^2]F = 0 \quad , \tag{3.12}$$

where prime denotes differentiation.

For the special case of $\delta = 2$, Eq. (3.12) is immediately integrable to the following Riccati equation:

$$2(\xi^2 - 1)F' + 4\xi\, F - F^2 = K \quad , \tag{3.13}$$

where K is an arbitrary constant. This equation may be converted into a linear, second-order, ordinary differential equation by the following transformation:

$$(\xi^2 - 1)^r \, V(z) = \mathrm{Exp}\left[-\frac{1}{2}\int^{\xi} \frac{F(\xi')}{\xi'^2-1}\, d\xi'\right] \quad , \tag{3.14}$$

$$2z = \xi + 1 \quad . \tag{3.15}$$

The result is:

$$z(1 - z)V'' + 2(1 + r)\,(1 - 2z)V' - (2r - K)V = 0 \quad , \tag{3.16}$$

where r satisfies the quadratic equation,

$$4r^2 + 4r + K = 0 \quad . \tag{3.17}$$

Equation (3.16) has three regular singular points at $z = 0$, 1, and ∞. The solutions to this equation are expressible in terms of hypergeometric functions. For $\xi > 1$, the appropriate general solution to (3.16) is, in the usual notation,

$$V(z) = Cz^{-2r-3} \; {}_2F_1\,[2r + 3,\, 2,\, 4;\; 1/z] \quad , \tag{3.18}$$

where C is an arbitrary constant. Thus, from Eq. (3.14), the corresponding expression for $F(\xi)$ is

$$F(\xi) = 6(\xi - 1) - 4r$$

$$+ 2(2r + 3)\,\frac{(\xi - 1)\; {}_2F_1\,[2r + 4,\, 3,\, 5;\; 2/(\xi + 1)]}{(\xi + 1)\; {}_2F_1\,[2r + 3,\, 2,\, 4;\; 2/(\xi + 1)]} \quad . \tag{3.19}$$

The expression (3.19) for $F(\xi)$ assumes some particularly simple forms in terms of elementary functions for special values of K. As examples, typical expressions for $F(\xi)$ and $\sigma(x, t)$ for two different values of K are listed below:

<u>K = 0, (i.e., r = 0, or - 1)</u>

$$F(\xi) = 8 \left\{2\xi + (\xi^2 - 1) \ln [(\xi - 1)/(\xi + 1)] + A_1 (\xi^2 - 1)\right\}^{-1} \quad . \quad (3.20)$$

$$\sigma(x, t) = 8x \left\{2xt + (t^2 - x^2) \ln [(t - x)/(t + x)] + A_1 (t^2 - x^2)\right\}^{-1} \quad . \quad (3.21)$$

<u>K = -3, (i.e., r = 1/2 or - 3/2)</u>

$$F(\xi) = 6 (1 + A_2 \xi + \xi^2)/(A_2 + 3\xi - \xi^2) \quad , \quad (3.22)$$

$$\sigma(x, t) = 6 (x^2 + A_2 xt + t^2)/(A_2 x^2 + 3x^2 t - t^3) \quad . \quad (3.23)$$

In these expressions, A_1, A_2, and A_3 are arbitrary constants.

It is interesting to note that the solution given by (3.21) is invariant under the translation defined by $(x', t') = (x + a, t + a)$, where a is an arbitrary constant. This property will be utilized in Section V to derive a closed form solution of a self-similar, unsteady, dispersed, nonlinear wave motion behind a constant "elastic-speed" shockfront propagating into an initially quiescent region.

IV. NONLINEAR WAVES WITH CONSTANT SPEEDS

Equation (2.7) is a nonlinear, hyperbolic differential equation of the evolution type. The characteristic speeds related to this equation are given by,

$$D_{\pm}\bar{X}/D\bar{t} = \pm c \quad , \tag{4.1}$$

or,

$$D_{\pm}x/Dt = \pm 1 \quad , \tag{4.2}$$

where $D_{\pm}(\cdot)/D\bar{t}$ and $D_{\pm}(\cdot)/Dt$ denote differentiation along the characteristics.

Due to the presence of the evolution or dissipative term, $[1_{+}(\sigma)\, df(\sigma)/d\sigma + \delta(\sigma)\, f(0)]\, \partial\sigma/\partial t$, in Eq. (2.7), it is expected that, in the plastic range, the material can also support dissipative, dispersive waves in addition to the characteristic propagations of discontinuities given by Eq. (4.1) or (4.2). To demonstrate the existence of non-characteristic propagations, a class of constant speed solutions to Eq. (2.7) is considered in this section. This class of solutions is obtained by searching for expressions of the form:

$$\sigma(x, t) = g(s) \quad , \tag{4.3}$$

where,

$$s \equiv \bar{c}t - x \quad , \tag{4.4}$$

and $\bar{c}$ = constant determines the speed of propagation.

Substituting Eq. (4.3) into Eq. (2.7), a nonlinear, second-order, ordinary differential equation results:

$$\beta(1 - \bar{c}^2)g'' = \bar{c}\,[1_+(g)\, f'(g) + \delta(g)\, f(0)]g' \quad , \tag{4.5}$$

where primes denote differentiation. This equation can be integrated once immediately to yield,

$$\beta(1 - \bar{c}^2)g' = \bar{c}\; 1_+(g)\, f(g) + A \quad , \tag{4.6}$$

where A is an arbitrary constant. Actually, the fact that Eq. (4.5) can be integrated once in closed form is not due to the special choice of the constitutive equation, (2.6), since, by assuming solutions of constant speeds of propagation, Eq. (2.2) can be integrated at once without making any additional assumptions. By comparing the expression (4.6) with the basic equations, (2.2) and (2.6), it is easily demonstrated that $A = 0$. Thus,

$$\beta(1 - \bar{c}^2)g' = \bar{c}\; 1_+(g)\, f(g) \quad , \tag{4.7}$$

for constant speeds of propagation.

If $g < 0$, then Eq. (4.7) becomes,

$$(1 - \bar{c}^2)g' = 0 \quad . \tag{4.8}$$

Except for the trivial case of g = constant, Eq. (4.8) requires that $\bar{c} = \pm 1$, which of course is the elastic speed of propagation. For $g > 0$, Eq. (4.7) requires that

$$\text{(i)} \quad \bar{c}^2 < 1, \quad \text{for } (g'/\bar{c}) > 0 \quad , \tag{4.9}$$

$$\text{(ii)} \quad \bar{c}^2 > 1, \quad \text{for } (g'/\bar{c}) < 0 \quad . \tag{4.10}$$

Thus, for the physically more meaningful case of $\bar{c}^2 < 1$, the overstress may increase or decrease with s depending on whether $\bar{c} > 0$ or < 0. Equations (4.9) and (4.10) also indicate that if solutions for $g > 0$ exist, the waves represented by these solutions are not characteristic propagations.

On setting $h = g'$, Eq. (4.5) for $g > 0$ may be written as

$$\frac{h'}{g'} = \frac{\bar{c}}{\beta(1 - \bar{c}^2)} f'(g) \quad . \tag{4.11}$$

Since f(g) is a C^1 function, Eq. (4.11) does not have any singular points. Thus, according to Poincaré-Bendixon theorem, it may be concluded that Eq. (4.11), in general, does not possess periodic solutions.

For $g > 0$, Eq. (4.7) may be integrated, in general, by quadrature as follows:

$$s = [\beta(1 - \bar{c}^2)/\bar{c}] \int^{g} f^{-1}(\zeta)\, d\zeta + C' \quad , \tag{4.12}$$

where C' is an arbitrary constant.

Perzyna [9] in 1963, suggested two interesting expressions for $f(\zeta)$. In slightly generalized forms, these expressions are given as follows:

$$\text{(i)} \quad f(\zeta) = \sum_{\ell=0}^{L} a_\ell\, \zeta^\ell \quad , \tag{4.13}$$

$$\text{(ii)} \quad f(\zeta) = b_o + \sum_{\ell=1}^{L} b_\ell \, (\operatorname{Exp} \zeta^\ell - 1) \quad , \tag{4.14}$$

where a_ℓ, b_ℓ are constants. Expression (i) or Eq. (4.13) includes the model $f(\zeta) = \zeta^\delta$ suggested by Cowper & Symonds [8] as a special case. For $F(\zeta) = \zeta^\delta$, and $\delta \neq 1$, Eq. (4.12) becomes,

$$s - s_1 = \frac{\delta(1 - \bar{c}^2)}{(1 - \delta)\bar{c}} \left(g^{1-\delta} - g_1^{\;1-\delta}\right) \quad , \tag{4.15}$$

where $g_1 = g(s_1)$, s_1 is a constant, and β has been chosen as b_1.

Another simple result is obtained for the constitutive relation (ii) or Eq. (4.14) with L = 1. The integrated expression is

$$s - s = \frac{(1 - \bar{c}^2)}{\bar{c}} \left[(g_1 - g) + \ell n\left(\frac{e^g - 1}{e^{g_1} - 1}\right) \right] \quad , \tag{4.16}$$

where, again, $g_1 = g(s_1)$, s_1 is a constant, and β has been chosen as b_1.

It is of interest to note that for the constitutive relation (i) or Eq. (4.13), the integral in Eq. (4.12) can always be evaluated in closed form. It is known that any real polynomial can be expressed as a product of factors, of which typical terms are $(\alpha_1\zeta + \alpha_2)^k$ and $(\beta_1\zeta^2 + 2\beta_2\zeta + \beta_3)^p$, where α_1, α_2, β_1, β_2, β_3 are real constants, $\beta_2^{\;2} < \beta_1\beta_3$ and k,p are real positive integers. Thus, $f^{-1}(\zeta)$ may be expressed in the form:

$$f^{-1}(\zeta) = \sum_{k=1}^{m} c_k \, (\alpha_1\zeta + \alpha_2)^{-k} + \sum_{p=1}^{n} (d_p\zeta + e_p)(\beta_1\zeta^2 + 2\beta_2\zeta + \beta_3)^{-p} \quad , \tag{4.17}$$

where c_k, d_p, and e_p are real constants.

The integrals from the first summation are of the type:

$$\int (\alpha_1 \zeta + \alpha_2)^{-k} \, d\zeta \quad , \tag{4.18}$$

which gives,

$$\begin{cases} \alpha_1^{-1} (1 - k)^{-1} (\alpha_1 \zeta + \alpha_2)^{1-k} \quad , \quad (k \neq 1) \quad , \\ \alpha_1^{-1} \ln |\alpha_1 \zeta + \alpha_2| \quad , \quad (k = 1) \quad . \end{cases} \tag{4.19}$$

The integrals from the second summation are of the type:

$$\begin{aligned} &\int (d_p \zeta + e_p) (\beta_1 \zeta^2 + 2\beta_2 \zeta + \beta_3)^{-p} \, d\zeta \\ &\quad = (d_p/\beta_1) \int (\beta_1 \zeta + \beta_2) (\beta_1 \zeta^2 + 2\beta_2 \zeta + \beta_3)^{-p} \, d\zeta \\ &\qquad + (e_p - \beta_2 d_p/\beta_1) \int (\beta_1 \zeta^2 + 2\beta_2 \zeta + \beta_3)^{-p} \, d\zeta \quad . \end{aligned} \tag{4.20}$$

The first integral in Eq. (4.20) gives:

$$\begin{cases} 2^{-1} (1 - p)^{-1} (\beta_1 \zeta^2 + 2\beta_2 \zeta + \beta_3)^{1-p} \quad , \quad (p \neq 1) \quad , \\ 2^{-1} \ln |\beta_1 \zeta^2 + 2\beta_2 \zeta + \beta_3| \quad , \quad (p = 1) \quad . \end{cases} \tag{4.21}$$

The second integral can be reduced as follows:

$$\int (\beta_1\zeta^2 + 2\beta_2\zeta + \beta_3)^{-p}\, d\zeta = \beta_1^{p-1} (\beta_1\beta_3 - \beta_2{}^2)^{1/2-p} \int (y^2 + 1)^{-p}\, dy \quad , \tag{4.22}$$

and the integral,

$$I_p = \int (y^2 + 1)^{-p}\, dy \quad , \tag{4.23}$$

can be evaluated from the recursion relation:

$$2(p - 1)\, I_p - (2p - 3)\, I_{p-1} = y(y^2 + 1)^{1-p} \quad , \tag{4.24}$$

with $I_1 = \tan^{-1} y$. The entire integral in Eq. (4.12) can, therefore, be evaluated in closed form in terms of elementary functions.

V. NONLINEAR WAVE MOTION BEHIND PROPAGATING DISCONTINUITIES

5.1. Self-similar solution behind a constant-speed shockfront.

Consider a one-dimensional shockfront propagating at some speed $U(>0)$ into an initially quiescent one-dimensional region ($x \geq 0$). As it had been remarked earlier, if the range of the rate of straining within the shock layer renders the material to exhibit instantaneous linear elasticity, then the shockfront will propagate at a constant speed,

$$U = \sqrt{E/\rho} \quad , \tag{5.1}$$

where E is the instantaneous modulus of elasticity. If the value of E is chosen to be the same as the modulus of the elastic range of the constitutive equation, (2.6), then the shock speed has the same value as the characteristic

speed $D_+\bar{X}/D\bar{t}$ given by Eq. (4.1). Under such an assumption, the values of u and $\bar{\sigma}$ immediately behind the shock layer must satisfy the characteristic compatibility condition [10]:

$$d\bar{\sigma} - \rho c du = - E\gamma f(\bar{\sigma}/\sigma_o - 1)1_+(\sigma/\sigma_o - 1)\ d\bar{t} \quad , \tag{5.2}$$

where $c = U = \sqrt{E/\rho}$.

The values of u, $\bar{\sigma}$, and ε, immediately behind the shock layer must also satisfy the jump conditions given by Eqs. (2.13) to (2.15). Since u, $\bar{\sigma} = 0$ in the quiescent region in front of the shock layer, Eq. (2.14) requires that,

$$\bar{\sigma} = - \rho U u = - \rho c u \quad , \tag{5.3}$$

immediately behind the shock layer. Combining Eqs. (5.2) and (5.3), the following differential equation results:

$$2\beta d\sigma = - f(\sigma)1_+(\sigma)\ dt \quad , \tag{5.4}$$

where as before, $\sigma \equiv (\bar{\sigma}/\sigma_o - 1)$ and $t \equiv \alpha c\bar{t}$ with $\alpha \equiv \beta\gamma\sqrt{\rho E}/\sigma_o$. Equation (5.4) indicates that, if $\sigma > 0$, then the overstress immediately behind the shockfront always attenuates with time along the shock.

Assuming that $\sigma > 0$ behind the shockfront, Eq. (5.4) can be integrated by quadrature as follows:

$$t = 2\beta \int^{\sigma} f^{-1}(\zeta)\ d\zeta + C'' \quad , \tag{5.5}$$

where the integral is identical to that of Eq. (4.12). Thus, closed-form solutions of Eq. (5.5) are possible for special constitutive assumptions.

For $f(\zeta) = \zeta^{\delta}$ with $\delta \neq 1$, Eq. (5.5) becomes

$$t = [2\delta/(1 - \delta)] \left(\sigma_1{}^{1-\delta} - \sigma^{1-\delta}\right) \quad , \tag{5.6}$$

where $\sigma_1 \equiv \sigma(0)$, and β has been chosen as δ. Therefore, the overstress immediately behind the shock attenuates monotonically with time along the shockfront from $\sigma = \sigma_1$ at $t = 0$ to $\sigma = 0$ at $t = \infty$.

It is interesting to note that for $\delta = 2$, Eq. (5.6) can be satisfied by one of the self-similar solutions given in Section III:

$$8\sigma^{-1} = (x + a) \left[2\eta + (\eta^2 - 1) \left\{A_1 + \ell n \, [(\eta - 1)/(\eta + 1)]\right\}\right] \quad , \tag{5.7}$$

where $\eta = (t + a)/(x + a)$, and a, A_1 are constants. To satisfy the compatibility condition (5.6) for $\delta = 2$, the constant a in (5.7) must be chosen as follows:

$$a = 4/\sigma_1 \quad , \tag{5.8}$$

Therefore, the dimensionless overstress $\sigma(x, t)$ behind the shockfront is given by,

$$\sigma = 8/\left[(t + 4/\sigma_1) \left\{2 + (\eta - 1/\eta) \, [A_1 + \ell n \, (\tfrac{r-1}{r+1})]\right\}\right] \quad , \tag{5.9}$$

and the stress boundary condition at x = 0 is,

$$\sigma(0, t) = 8/\left[2(t + 4/\sigma_1) + (2t + \sigma_1 t^2/4) \left\{A_1 - \ell n \, [1 + 8/(\sigma_1 t)]\right\}\right] \quad . \tag{5.10}$$

The behavior of this function for various values of A_1 is shown in Fig. 1.

In dimensional forms, the resulting expressions for $\sigma(\bar{X}, \bar{t})$, $u(\bar{X}, \bar{t})$, and $\varepsilon(\bar{X}, \bar{t})$ for this case are given as follows:

$$\sigma = \bar{\sigma}/\sigma_o - 1$$

$$= [4c\, \sigma_o/(E\gamma)]\, (\bar{X} + ct_o)$$

$$\Big[2c\, (\bar{X} + ct_o)\, (\bar{t} + t_o) - [c^2\bar{t}\, (\bar{t} + 2t_o) - \bar{X}\, (\bar{X} + 2ct_o)]$$

$$\{\ell n\, [(c\bar{t} + \bar{X} + 2ct_o)/(c\bar{t} - \bar{X})] + B\}\Big]^{-1} \quad , \tag{5.11}$$

$$u = -\, (\sigma_o/\rho)\, [\sigma(\bar{t} + t_o)/(\bar{X} + ct_o) + c^{-1}] \quad , \tag{5.12}$$

$$\varepsilon = (\sigma_o/\rho)\, [\sigma(\bar{t} + t_o)^2/(\bar{X} + ct_o)^2 + c^{-2}] \quad , \tag{5.13}$$

where

$$t_o = 2\sigma_o/(\gamma\sigma_1 E) \quad , \tag{5.14}$$

and B is a constant. The behavior of the functions $u(0, \bar{t})$ and $\varepsilon(0, \bar{t})$ for various values of A_1 are indicated in Figs. 2 and 3. It is of interest to note that (5.13) yields a permanent strain ε_p given by

$$\varepsilon_p = \lim_{\bar{t}\to\infty} \varepsilon = -\, (\sigma_o/E)\, [(4c\, \sigma_o/\beta\gamma E)\, (\bar{X} + ct_o)^{-1} - 1] \quad . \tag{5.15}$$

5.2. Constant speed solution behind an elastic precurser.

Duvall [22] suggested that the one-dimensional, unsteady motion in a semi-infinite ($x \geq 0$), rate-sensitive, elastoplastic region generated by a continuously applied load at its boundary ($x = 0$) may eventually consist of an elastic precurser propagating into an initially quiescent region, an unloading, relaxation zone, and a sub-elastic, constant-speed, nonlinear wave motion as depicted in Fig. 4. After a reasonable length of time, the elastic precurser will be far ahead of the leading wave of the constant speed region and the boundary of $x = 0$ will be far behind it. Relative to an observer moving with the leading wave of the constant speed region, the unsteady motion behind the leading wave becomes essentially steady.

Any of the constant-speed, nonlinear wave solutions described by Eq. (4.12) in Sec. IV with $\bar{c} < 1$ may be considered as a constant-speed portion of such an "unsteady" motion. If the overstress on the leading wave $s = s_1$ is $\sigma_1 > 0$, then the quadrature expression, (4.12), becomes,

$$s - s_1 = [\beta(1 - \bar{c}^2)/\bar{c}] \int_{\sigma_1}^{\sigma} f^{-1}(\zeta)\, d\zeta \quad , \tag{5.16}$$

where $s \equiv \bar{c}t - x$, and $\bar{c} < 1$. From Eq. (5.16), the stress boundary condition required to maintain the constant-speed motion can be evaluated in a straightforward manner.

It is of interest to note that, if the initial rate of loading of the applied stress at the boundary is not too high such that the time required to raise the stress $\bar{\sigma}$ from zero to the value of the static yield stress σ_o is

much longer than the pertinent characteristic relaxation time of the medium, then a complete description of a possible nonlinear wave motion for a continuously loading boundary may be constructed exactly. Figure 5 is a schematic representation of such a motion. It consists of four solution regions separated by three discontinuities described as follows:

(1) Solution Regions

R_1: the undisturbed region, $\bar{\sigma} = 0$

R_2: the elastic region, $\bar{\sigma} < \sigma_o$

R_3: the constant stress region, $\bar{\sigma} = \sigma_o$

R_4: the constant speed solution region, $\bar{\sigma} > \sigma_o$.

(2) Discontinuities

s_1: the leading elastic wave

s_2: the trailing elastic wave

s_3: the leading constant speed wave.

Such an unsteady motion may be generated by a monotonically increasing stress boundary condition. The manner in which the stress varies at the boundary in the elastic range can be quite arbitrary (so long as the rate of loading is small enough so that there will be no dynamic overstressing in the elastic precurser) and has been chosen as a linear function of $\bar{t}$ in Fig. 5 for simplicity, while the rate of stressing beyond the static yield stress must follow the expression given in Eq. (4.12) or the differential equation, (4.7).

It is clear from Eq. (4.7) that nontrivial solutions in R_4 can be generated from a leading wave s_3 on which the stress is $\bar{\sigma} = 0$ only if $f(0) > 0$. The value of f(0) can be arbitrarily small. Alternatively, if $f(0) = 0$ for a specific constitutive relation, a solution such as the one depected by Fig. 5 can still be generated by viewing s_3 as a small discontinuity in the value of $\bar{\sigma}$ such that the overstress jumps across this "plastic shock" from zero to some small constant positive value $\sigma_1 << 1$. The jump in σ from 0 to σ_1 at the boundary of $\bar{X} = 0$ may be viewed as the result of a very fast rate of loading r in a small interval of time $\Delta\bar{t}$ near $\bar{t} = \bar{t}_o$ such that $\lim_{\Delta\bar{t}\to o} (r\,\Delta\bar{t}) = \bar{\sigma}_1 = (1 + \sigma_1)\sigma_o$ and $0 < \sigma_1 << 1$. The fact that $\bar{\sigma} = \sigma_o$ in R_3 is an admissible solution can be deduced immediately from Eq. (4.7).

REFERENCES

1. Rivlin, R. S., 1956, Q. Appl. Math., 14, 83.

2. Cristescu, N., 1967, "Dynamic Plasticity," North-Holland, Amsterdam.

3. Sokolovskii, V. V., 1948, Dokl. Akad. Nauk SSSR, 60, 775.

4. Sokolovskii, V. V., 1948, Prikl. Mat. Mech., 12, 261.

5. Pipkin, A. C., 1965, Q. Appl. Math., 23, 297.

6. Varley, E. and T. G. Rogers, 1967, Proc. Roy. Soc. (London), Ser. A, 296, 498.

7. Courant, R. and K. O. Friedrichs, 1948, "Supersonic Flow and Shock Waves," Interscience, New York.

8. Cowper, G. R. and P. S. Symonds, 1957, Technical Report No. 28, Contract Nonr 562 (10), Brown University, Providence, R. I.

9. Perzyna, P., 1963, Arch. Mech. Stos., 15, 113.

10. Malvern, L. E., 1951, Q. Appl. Math., 8, 405.

11. Coleman, B. D. and M. E. Gurtin, 1968, Proc. IUTAM Symp. Vienna, 54, Springer-Verlag, New York.

12. Dunwoody, J. and N. Dunwoody, 1965, Int. J. Engr. Sci., 3, 417.

13. Parker, D. F. and E. Varley, 1968, Q. J. Mech. Appl. Math., 21(3), 329.

14. Manjoine, M. J., 1944, J. Appl. Mech., 11, 211.

15. Parkes, E. W., 1958, Proc. Inst. Civil Engineers, 10, 277.

16. Bodner, S. R. and P. S. Symonds, 1960, Proc. 2nd Symp. Nav. Struc. Mech., 488, Pergamon Press, Oxford.

17. Ting, T. C. T. and P. S. Symonds, 1962, Proc. 4th U. S. Natl. Congr. Appl. Mech., 1153.

18. Bodner, S. R. and P. S. Symonds, 1962, J. Appl. Mech., 29, 719.

19. Ting, T. C. T., 1963, Q. Appl. Math. 21, 133.

20. Perrone, N., 1965, J. Appl. Mech., 32, 489.

21. Morgan, A. J. A., 1952, Q. J. Math, 2, 250.

22. Duvall, G. E., 1964, Proc. IUTAM Symp. Stress Waves in Anelastic Solids, 20, Springer-Verlag, New York.

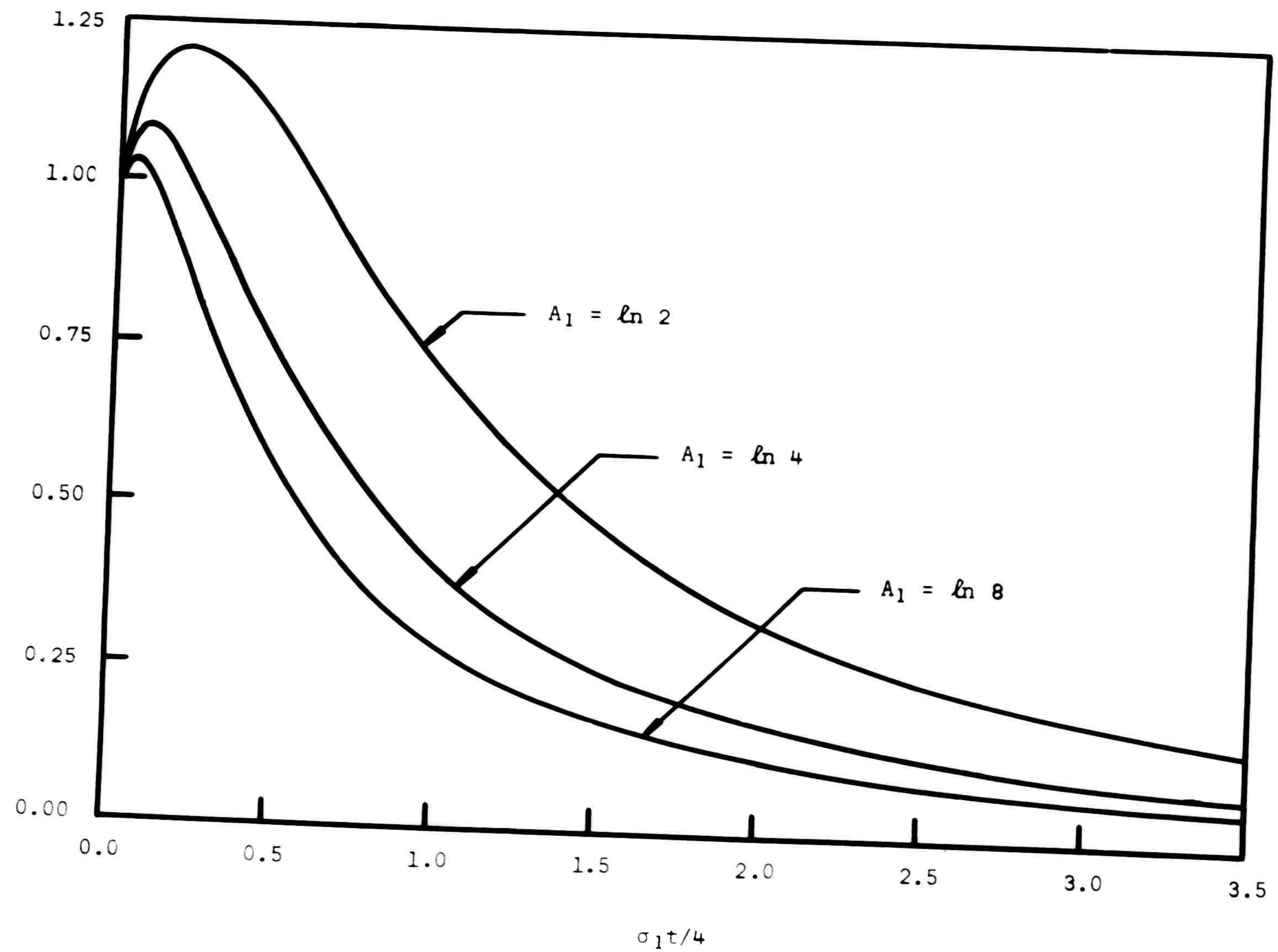

Figure 1. Variation of σ/σ_1 with $\sigma_1 t/4$ at $\overline{X} = 0$.

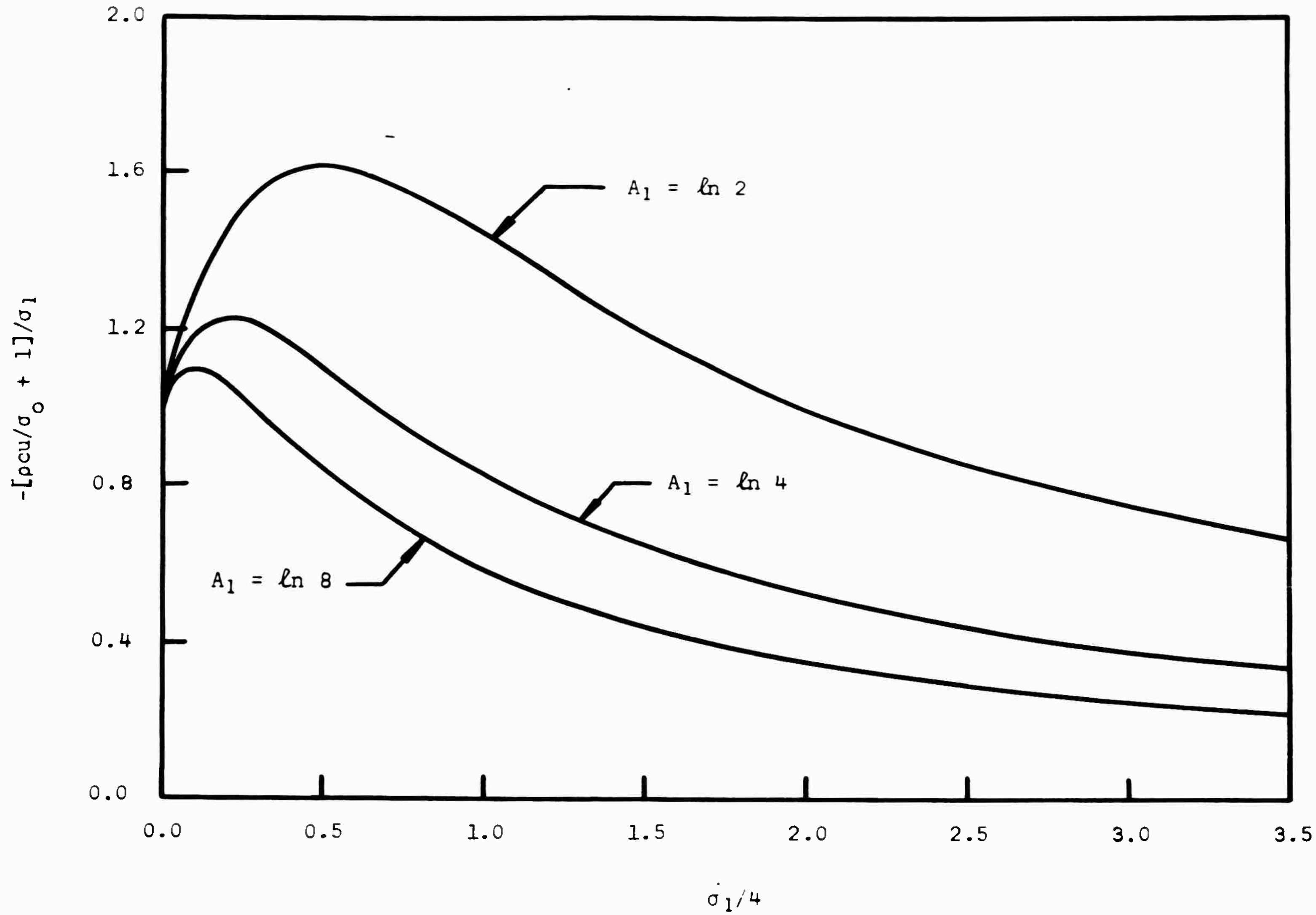

Figure 2. Variation of $-[\rho cu/\sigma_o + 1]/\sigma_1$ with $\sigma_1 t/4$ at $\overline{X} = 0$.

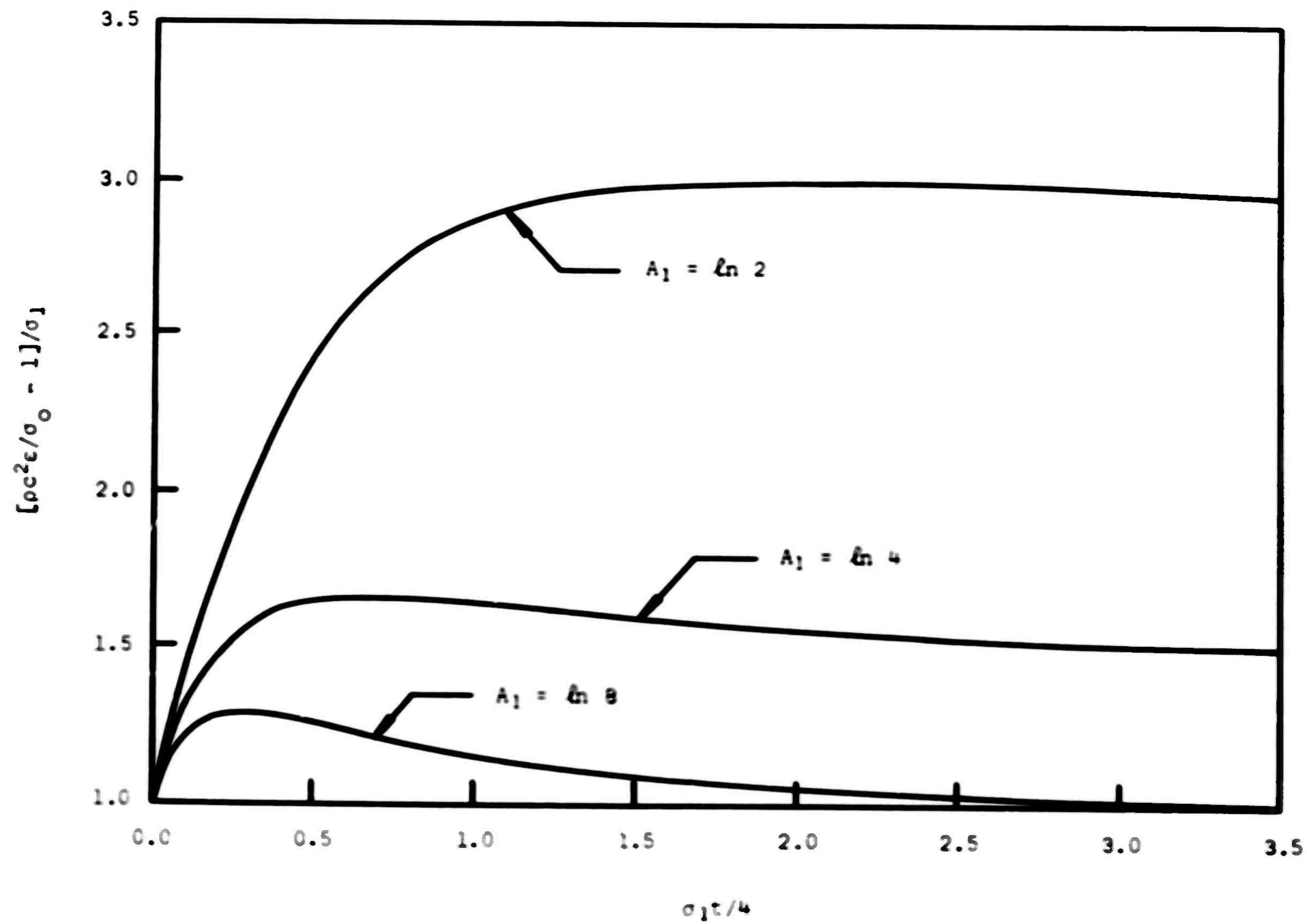

Figure 3. Variation of $[\rho c^2\varepsilon/\sigma_o - 1]/\sigma_1$ with $\sigma_1 t/4$ at $\overline{X} = 0$.

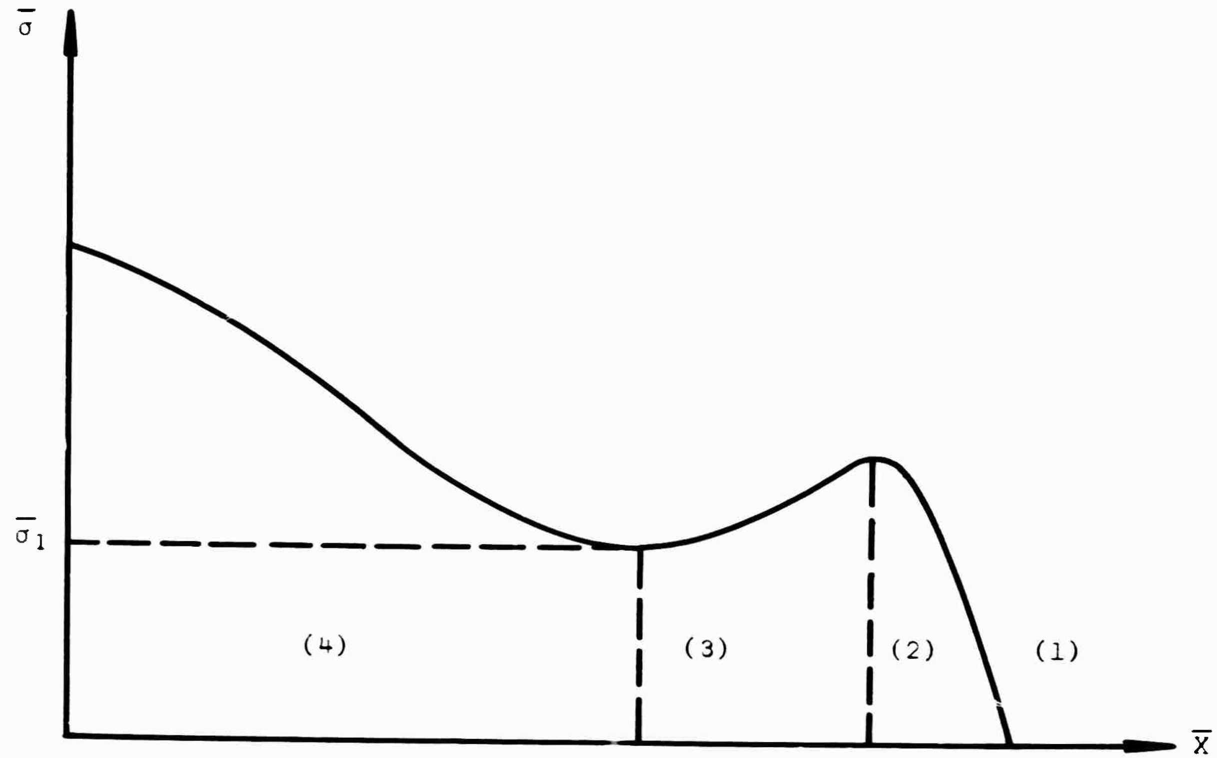

Figure 4. Schematic representation of a nonlinear wave motion behind a relaxation zone. (1) quiescent region, (2) elastic precurser, (3) relaxation zone, (4) sub-elastic, constant speed region.

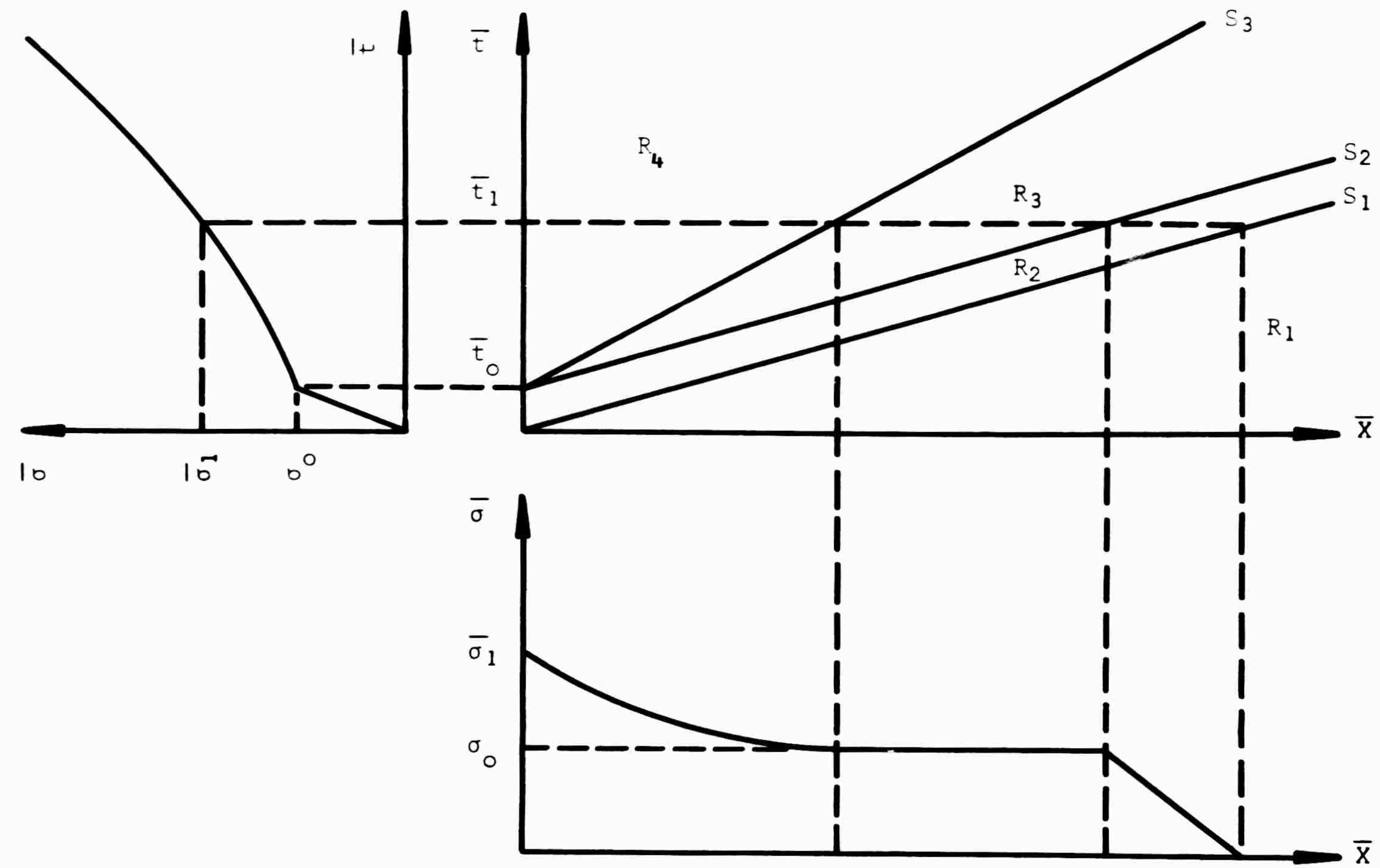

Figure 5. Schematic representation of a possible nonlinear wave motion due to a continuously loading boundary at $\bar{X} = 0$.

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) North Carolina State University Raleigh, North Carolina	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Some One-Dimensional Solutions of Nonlinear Waves of a Rate-Sensitive, Elastoplastic Material

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

Burniston, E. E.
Chang, T. S.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January 1970	30 + 4	22

8a. CONTRACT OR GRANT NO. N00014-68-A-0187 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 70-1 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Qualified requesters may obtain copies of this report from DDC.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Advanced Research Projects Agency Washington, D. C. 20301

13. ABSTRACT

Two classes of closed form solutions of one-dimensional, nonlinear waves of a rate-sensitive, elastoplastic material are reported. One class of these solutions is self-similar and the other class consists of constant speed propagations. Applications of these solutions to unsteady motions behind propagating discontinuities are also considered.

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
One-dimensional motion						
Nonlinear waves						
Rate-sensitive, elastoplastic material						
Closed form solutions						

DD FORM 1 NOV 65 1473 (BACK)
S/N 0101-807-6821

Unclassified
Security Classification

A-31409